

# CHARACTER AMENABILITY AND CHARACTER INNER AMENABILITY OF MORPHISM PRODUCT OF BANACH ALGEBRAS

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**ABSTRACT.** Let  $T$  be a Banach algebra homomorphism from a Banach algebra  $\mathcal{B}$  to a Banach algebra  $\mathcal{A}$  with  $\|T\| \leq 1$ . Recently it has been obtained some results about Arens regularity and also various notions of amenability of  $\mathcal{A} \times_T \mathcal{B}$ , in the case where  $\mathcal{A}$  is commutative. In the present paper, most of these results have been generalized and proved for an arbitrary Banach algebra  $\mathcal{A}$ .

## 1. Introduction

Let  $\mathcal{A}$  and  $\mathcal{B}$  be two Banach algebras and let  $T \in \text{hom}(\mathcal{B}, \mathcal{A})$ , the set consisting of all Banach algebra homomorphisms from  $\mathcal{B}$  into  $\mathcal{A}$  with  $\|T\| \leq 1$ . Following [1] and [2], the Cartesian product space  $\mathcal{A} \times \mathcal{B}$  equipped with the following algebra multiplication

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2 + a_1 T(b_2) + T(b_1) a_2, b_1 b_2), \quad (a_1, a_2 \in \mathcal{A}, b_1, b_2 \in \mathcal{B}) \quad (1.1)$$

and the norm

$$\|(a, b)\| = \|a\|_{\mathcal{A}} + \|b\|_{\mathcal{B}},$$

is a Banach algebra which is denoted by  $\mathcal{A} \times_T \mathcal{B}$ . Note that  $\mathcal{A}$  is a closed ideal of  $\mathcal{A} \times_T \mathcal{B}$  and  $(\mathcal{A} \times_T \mathcal{B})/\mathcal{A}$  is isometrically isomorphic to  $\mathcal{B}$ . As we mentioned in [1], our definition of the multiplication  $\times_T$ , is presented by a slight difference with that given by Bhatt and Dabhi [2]. In fact they give the definition under the assumption of commutativity of  $\mathcal{A}$ . However this assumption is redundant, and the definition can be provided as (1.1), for an arbitrary Banach algebra  $\mathcal{A}$ .

Suppose that  $\mathcal{A}$  is unital with the unit element  $e$  and  $\varphi : \mathcal{B} \rightarrow \mathbb{C}$  is a multiplicative continuous linear functional. Define  $\theta : \mathcal{B} \rightarrow \mathcal{A}$  by  $\theta(b) = \varphi(b)e$  ( $b \in \mathcal{B}$ ). As it is mentioned in [2], the above introduced product  $\times_\theta$  with respect to  $\theta$ , coincides with  $\theta$ -Lau product of  $\mathcal{A}$  and  $\mathcal{B}$ , investigated by Lau [5] for the certain classes of Banach algebras. This definition was extended by M. Sangani Monfared [6], for the general case.

Bhatt and Dabhi [2] studied Arens regularity and amenability of  $\mathcal{A} \times_T \mathcal{B}$ . In fact, they proved that Arens regularity as well as amenability (together with its various avatars) of  $\mathcal{A} \times_T \mathcal{B}$  are stable with respect to  $T$ . Moreover, in a recent work [1], we verified biprojectivity and biflatness of  $\mathcal{A} \times_T \mathcal{B}$ , with respect to our definition (1.1), and showed that both are stable with respect to  $T$ . Finally, as an application of these results, we obtained that  $\mathcal{A} \times_T \mathcal{B}$  is amenable (respectively, contractible) if

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and only if both  $\mathcal{A}$  and  $\mathcal{B}$  so are. In fact we proved [2, Theorem 4.1, Part (1)] for the case where  $\mathcal{A}$  is not necessarily commutative.

The aim of the present work is investigating the results of [2], with respect to our definition of  $\times_T$ , and in fact whenever  $\mathcal{A}$  and  $\mathcal{B}$  are arbitrary Banach algebras. We first study the relation between left (right) topological centers  $(\mathcal{A} \times_T \mathcal{B})''$ ,  $\mathcal{A}''$  and  $\mathcal{B}''$ , and as an important result we prove that if  $T$  is epimorphism, then Arens regularity of  $(\mathcal{A} \times_T \mathcal{B})$  is stable with respect to  $T$ . In fact we prove that  $(\mathcal{A} \times_T \mathcal{B})$  is Arens regular if and only if both  $\mathcal{A}$  and  $\mathcal{B}$  so are. Moreover, we investigate some of the known results about  $\theta$ -Lau product of the Banach algebras  $\mathcal{A}$  and  $\mathcal{B}$ , given in [7, Proposition 2.8], for the morphism product  $\mathcal{A} \times_T \mathcal{B}$ . Furthermore we study some notions of amenability for  $\mathcal{A} \times_T \mathcal{B}$ , which have been studied in [2]. We show that weak amenability as well as character amenability and also character inner amenability of  $\mathcal{A} \times_T \mathcal{B}$  are stable with respect to  $T$ . In fact we prove that  $\mathcal{A} \times_T \mathcal{B}$  is weakly amenable (respectively, character amenable, character inner amenable) if and only if both  $\mathcal{A}$  and  $\mathcal{B}$  so are. All of these results are generalizations of those, discussed in [2].

## 2. Preliminaries

Let  $\mathcal{A}$  and  $\mathcal{B}$  be Banach algebras and  $T \in \text{hom}(\mathcal{B}, \mathcal{A})$ . Let  $\mathcal{A}'$  and  $\mathcal{A}''$  be the dual and second dual Banach spaces, respectively. Let  $a \in \mathcal{A}$ ,  $f \in \mathcal{A}'$  and  $\Phi, \Psi \in \mathcal{A}''$ . Then  $f \cdot a$  and  $a \cdot f$  are defined as  $f \cdot a(x) = f(ax)$  and  $a \cdot f(x) = f(xa)$ , for all  $x \in \mathcal{A}$ , making  $\mathcal{A}'$  an  $\mathcal{A}$ -bimodule. Moreover for all  $f \in \mathcal{A}'$  and  $\Phi \in \mathcal{A}''$ , we define  $\Phi \cdot f$  and  $f \cdot \Phi$  as the elements  $\mathcal{A}'$  by

$$\langle \Phi \cdot f, a \rangle = \langle \Phi, f \cdot a \rangle \text{ and } \langle f \cdot \Phi, a \rangle = \langle \Phi, a \cdot f \rangle \quad (a \in \mathcal{A}).$$

This defines two Arens products  $\square$  and  $\diamond$  on  $\mathcal{A}''$  as

$$\langle \Phi \square \Psi, f \rangle = \langle \Phi, \Psi \cdot f \rangle \text{ and } \langle \Phi \diamond \Psi, f \rangle = \langle \Psi, f \cdot \Phi \rangle,$$

making  $\mathcal{A}''$  a Banach algebra with each. The products  $\square$  and  $\diamond$  are called respectively, the first and second Arens products on  $\mathcal{A}''$ . Note that  $\mathcal{A}$  is embedded in its second dual via the identification

$$\langle a, f \rangle = \langle f, a \rangle \quad (f \in \mathcal{A}').$$

Also for all  $a \in \mathcal{A}$  and  $\Phi \in \mathcal{A}''$ , we have

$$a \square \Phi = a \diamond \Phi \quad \text{and} \quad \Phi \square a = \Phi \diamond a.$$

The left and right topological centers of  $\mathcal{A}''$  are defined as

$$\mathcal{Z}_t^{(\ell)}(\mathcal{A}'') = \{\Phi \in \mathcal{A}'' : \Phi \square \Psi = \Phi \diamond \Psi, (\Psi \in \mathcal{A}'')\}$$

and

$$\mathcal{Z}_t^{(r)}(\mathcal{A}'') = \{\Phi \in \mathcal{A}'' : \Psi \square \Phi = \Psi \diamond \Phi, (\Psi \in \mathcal{A}'')\}.$$

The algebra  $\mathcal{A}$  is called Arens regular if these products coincide on  $\mathcal{A}''$ ; or equivalently  $\mathcal{Z}_t^{(\ell)}(\mathcal{A}'') = \mathcal{Z}_t^{(r)}(\mathcal{A}'') = \mathcal{A}''$ .

Now consider  $\mathcal{A} \times_T \mathcal{B}$ . As we mentioned in [1], the dual space  $(\mathcal{A} \times_T \mathcal{B})'$  can be identified with  $\mathcal{A}' \times \mathcal{B}'$ , via the linear map  $\theta : \mathcal{A}' \times \mathcal{B}' \rightarrow (\mathcal{A} \times_T \mathcal{B})'$ , defined by

$$\langle (a, b), \theta((f, g)) \rangle = \langle a, f \rangle + \langle b, g \rangle,$$

where  $a \in \mathcal{A}, f \in \mathcal{A}', b \in \mathcal{B}$  and  $g \in \mathcal{B}'$ . Moreover,  $(\mathcal{A} \times_T \mathcal{B})'$  is a  $(\mathcal{A} \times_T \mathcal{B})$ -bimodule with natural module actions of  $\mathcal{A} \times_T \mathcal{B}$  on its dual. In fact it is easily verified that

$$(f, g) \cdot (a, b) = (f \cdot a + f \cdot T(b), f \circ (L_a T) + g \cdot b)$$

and

$$(a, b) \cdot (f, g) = (a \cdot f + T(b) \cdot f, f \circ (R_a T) + b \cdot g),$$

where  $a \in \mathcal{A}, b \in \mathcal{B}, f \in \mathcal{A}'$  and  $g \in \mathcal{B}'$ . In addition,  $L_a T : \mathcal{B} \rightarrow \mathcal{A}$  and  $R_a T : \mathcal{B} \rightarrow \mathcal{A}$  are defined as  $L_a T(y) = aT(y)$  and  $R_a T(y) = T(y)a$ , for each  $y \in \mathcal{B}$ . Furthermore,  $\mathcal{A} \times_T \mathcal{B}$  is a Banach  $\mathcal{A}$ -bimodule under the module actions

$$c \cdot (a, b) := (c, 0) \cdot (a, b) \text{ and } (a, b) \cdot c := (a, b) \cdot (c, 0),$$

for all  $a, c \in \mathcal{A}$  and  $b \in \mathcal{B}$ . Also  $\mathcal{A} \times_T \mathcal{B}$  can be made into a Banach  $\mathcal{B}$ -bimodule in a similar fashion.

### 3. Arens regularity

Let  $\mathcal{A}$  and  $\mathcal{B}$  be Banach algebras and  $T \in \text{hom}(\mathcal{B}, \mathcal{A})$ . Define  $T' : \mathcal{A}' \rightarrow \mathcal{B}'$ , by  $T'(f) = f \circ T$  and  $T'' : \mathcal{B}'' \rightarrow \mathcal{A}''$ , as  $T''(F) = F \circ T'$ . Then by [3, Page 251], both

$$T'' : (\mathcal{B}'', \square) \rightarrow (\mathcal{A}'', \square)$$

and

$$T'' : (\mathcal{B}'', \diamond) \rightarrow (\mathcal{A}'', \diamond)$$

are continuous Banach algebra homomorphisms. Also in both the cases,  $\|T''\| \leq 1$ . Moreover if  $T$  is epimorphism, then so is  $T''$ . It is easy to obtain that  $T''(b) = T(b)$ , for each  $b \in \mathcal{B}$ .

In this section we investigate the results of the third section of [2], with respect to the definition (1.1), and for the case where  $\mathcal{A}$  is not necessarily commutative. We commence with the following proposition, which shows that Part (1) of [2, Theorem 3.1] is established, under our assumptions.

**Proposition 3.1.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be Banach algebras and  $T \in \text{hom}(\mathcal{B}, \mathcal{A})$ . Moreover suppose that  $\mathcal{A}'', \mathcal{B}''$  and  $(\mathcal{A} \times_T \mathcal{B})''$  are equipped with their first (respectively, second) Arens products. Then  $\mathcal{A}'' \times_{T''} \mathcal{B}'' \cong (\mathcal{A} \times_T \mathcal{B})''$ , as isometric isomorphism.*

*Proof.* Define

$$\Theta : \mathcal{A}'' \times_{T''} \mathcal{B}'' \rightarrow (\mathcal{A} \times_T \mathcal{B})''$$

by

$$\Theta(\Phi, \Psi)(f, g) = \Phi(f) + \Psi(g),$$

for all  $\Phi \in \mathcal{A}'', \Psi \in \mathcal{B}''$  and  $f \in \mathcal{A}'$  and  $g \in \mathcal{B}'$ . It is easy to obtain that  $\Theta$  is a bijective isometric linear map. We show that  $\Theta$  is an algebraic homomorphism with both Arens products. By (1.1), the product on  $\mathcal{A}'' \times_{T''} \mathcal{B}''$  with respect to the first Arens product is as

$$(\Phi_1, \Psi_1)(\Phi_2, \Psi_2) = (\Phi_1 \square \Phi_2 + \Phi_1 \square T''(\Psi_2) + T''(\Psi_1) \square \Phi_2, \Psi_1 \square \Psi_2),$$

where  $(\Phi_1, \Psi_1), (\Phi_2, \Psi_2) \in \mathcal{A}'' \times_{T''} \mathcal{B}''$ . We show that

$$\Theta((\Phi_1, \Psi_1)(\Phi_2, \Psi_2)) = \Theta(\Phi_1, \Psi_1) \square \Theta(\Phi_2, \Psi_2).$$

We compute the Arens product  $\square$  on  $(\mathcal{A} \times_T \mathcal{B})''$ . For all  $(a, b) \in \mathcal{A} \times_T \mathcal{B}$  and  $(f, g) \in \mathcal{A}' \times \mathcal{B}'$  we have

$$(f, g) \cdot (a, b) = (f \cdot a + f \cdot T(b), T'(f \cdot a) + g \cdot b)$$

Also for  $(\Phi, \Psi) \in \mathcal{A}'' \times_{T''} \mathcal{B}''$ ,  $(f, g) \in \mathcal{A}' \times \mathcal{B}'$  and all  $(a, b) \in \mathcal{A} \times_T \mathcal{B}$  we have

$$\begin{aligned} (\Theta(\Phi, \Psi) \cdot (f, g))(a, b) &= \Theta(\Phi, \Psi)((f, g) \cdot (a, b)) \\ &= \Theta(\Phi, \Psi)((f \cdot a + f \cdot T(b), T'(f \cdot a) + g \cdot b)) \\ &= \Phi(f \cdot a + f \cdot T(b)) + \Psi(T'(f \cdot a) + g \cdot b) \\ &= \Phi(f \cdot a) + \Phi(f \cdot T(b)) + T''(\Psi)(f \cdot a) + \Psi(g \cdot b) \\ &= (\Phi \cdot f + T''(\Psi) \cdot f)(a) + (\Phi \cdot f)(T(b)) + (\Psi \cdot g)(b) \\ &= (\Phi \cdot f + T''(\Psi) \cdot f)(a) + (T'(\Phi \cdot f) + \Psi \cdot g)(b) \\ &= (\Phi \cdot f + T''(\Psi) \cdot f, T'(\Phi \cdot f) + \Psi \cdot g)(a, b). \end{aligned}$$

Thus

$$\Theta(\Phi, \Psi) \cdot (f, g) = (\Phi \cdot f + T''(\Psi) \cdot f, T'(\Phi \cdot f) + \Psi \cdot g).$$

Consequently for  $(\Phi_1, \Psi_1), (\Phi_2, \Psi_2) \in \mathcal{A}'' \times_{T''} \mathcal{B}''$  and all  $(f, g) \in \mathcal{A}' \times \mathcal{B}'$  we obtain

$$\begin{aligned} [\Theta(\Phi_1, \Psi_1) \square \Theta(\Phi_2, \Psi_2)](f, g) &= \langle (\Theta(\Phi_1, \Psi_1), \Theta(\Phi_2, \Psi_2)) \cdot (f, g) \rangle \\ &= \langle \Theta(\Phi_1, \Psi_1), (\Phi_2 \cdot f + T''(\Psi_2) \cdot f, T'(\Phi_2 \cdot f) + \Psi_2 \cdot g) \rangle \\ &= \Phi_1(\Phi_2 \cdot f + T''(\Psi_2) \cdot f) + \Psi_1(T'(\Phi_2 \cdot f) + \Psi_2 \cdot g) \\ &= \Phi_1(\Phi_2 \cdot f + T''(\Psi_2) \cdot f) + T''(\Psi_1)(\Phi_2 \cdot f) + \Psi_1(\Psi_2 \cdot g) \\ &= (\Phi_1 \square \Phi_2 + \Phi_1 \square T''(\Psi_2) + T''(\Psi_1) \square \Phi_2)(f) + (\Psi_1 \square \Psi_2)(g) \\ &= (\Phi_1 \square \Phi_2 + \Phi_1 \square T''(\Psi_2) + T''(\Psi_1) \square \Phi_2, \Psi_1 \square \Psi_2)(f, g) \\ &= [(\Phi_1, \Psi_1)(\Phi_2, \Psi_2)](f, g), \end{aligned}$$

As claimed. Thus  $\Theta$  is an algebraic homomorphism with the first Arens product. Similarly one can show that for all  $(a, b) \in \mathcal{A} \times_T \mathcal{B}$ ,  $(f, g) \in \mathcal{A}' \times \mathcal{B}'$  and  $(\Phi, \Psi) \in \mathcal{A}'' \times_{T''} \mathcal{B}''$

$$(a, b) \cdot (f, g) = (a \cdot f + T(b) \cdot f, T'(a \cdot f) + b \cdot g)$$

and

$$(f, g) \cdot \Theta(\Phi, \Psi) = (f \cdot \Phi + f \cdot T''(\Psi), T'(f \cdot \Phi) + g \cdot \Psi).$$

Moreover for all  $(\Phi_1, \Psi_1), (\Phi_2, \Psi_2) \in \mathcal{A}'' \times_{T''} \mathcal{B}''$ , analogously we obtain

$$\begin{aligned} \Theta(\Phi_1, \Psi_1) \diamond \Theta(\Phi_2, \Psi_2) &= (\Phi_1 \diamond \Phi_2 + \Phi_1 \diamond T''(\Psi_2) + T''(\Psi_1) \diamond \Phi_2, \Psi_1 \diamond \Psi_2) \\ &= (\Phi_1, \Psi_1)(\Phi_2, \Psi_2), \end{aligned}$$

when we consider  $\mathcal{A}'' \times_{T''} \mathcal{B}''$  with the second Arens product. Consequently  $\Theta$  is also an algebraic homomorphism, with respect to the second Arens product.  $\square$

In the next theorem, we investigate Part (2) of [2, Theorem 3.1], for an arbitrary Banach algebra  $\mathcal{A}$ . We present our proof only for the left topological center. Calculations and results for the right version are analogous.

**Theorem 3.2.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be Banach algebras and  $T \in \text{hom}(\mathcal{B}, \mathcal{A})$ .*

- (i) *If  $(\Phi, \Psi) \in \mathcal{Z}_t^{(\ell)}((\mathcal{A} \times_T \mathcal{B})'')$ , then  $(\Phi + T''(\Psi), \Psi) \in \mathcal{Z}_t^{(\ell)}(\mathcal{A}'') \times_{T''} \mathcal{Z}_t^{(\ell)}(\mathcal{B}'')$ .*
- (ii) *If  $(\Phi, \Psi) \in \mathcal{Z}_t^{(\ell)}(\mathcal{A}'') \times_{T''} \mathcal{Z}_t^{(\ell)}(\mathcal{B}'')$ , then  $(\Phi - T''(\Psi), \Psi) \in \mathcal{Z}_t^{(\ell)}((\mathcal{A} \times_T \mathcal{B})'')$ .*

- (iii) If  $T$  is epimorphism, then  $\mathcal{Z}_t^{(\ell)}((\mathcal{A} \times_T \mathcal{B})'') = \mathcal{Z}_t^{(\ell)}(\mathcal{A}'') \times_{T''} \mathcal{Z}_t^{(\ell)}(\mathcal{B}'')$ . In particular,  $\mathcal{A} \times_T \mathcal{B}$  is Arens regular if and only if both  $\mathcal{A}$  and  $\mathcal{B}$  are Arens regular.

*Proof.* (i). Let  $(\Phi, \Psi) \in \mathcal{Z}_t^{(\ell)}((\mathcal{A} \times_T \mathcal{B})'')$ . Thus for each  $(\Phi', \Psi') \in (\mathcal{A} \times_T \mathcal{B})''$  we have

$$(\Phi, \Psi) \square (\Phi', \Psi') = (\Phi, \Psi) \diamond (\Phi', \Psi').$$

It follows that

$$\Phi \square \Phi' + \Phi \square T''(\Psi') + T''(\Psi) \square \Phi' = \Phi \diamond \Phi' + \Phi \diamond T''(\Psi') + T''(\Psi) \diamond \Phi' \quad (3.1)$$

and

$$\Psi \square \Psi' = \Psi \diamond \Psi'. \quad (3.2)$$

The equality (3.2) implies that  $\Psi \in \mathcal{Z}_t^{(\ell)}(\mathcal{B}'')$ . Moreover by choosing  $\Psi' = 0$  in (3.1) we obtain

$$(\Phi + T''(\Psi)) \square \Phi' = (\Phi + T''(\Psi)) \diamond \Phi', \quad (3.3)$$

for all  $\Phi' \in \mathcal{A}''$ , which implies that  $\Phi + T''(\Psi) \in \mathcal{Z}_t^{(\ell)}(\mathcal{A}'')$ . Consequently

$$(\Phi + T''(\Psi), \Psi) \in \mathcal{Z}_t^{(\ell)}(\mathcal{A}'') \times_{T''} \mathcal{Z}_t^{(\ell)}(\mathcal{B}'').$$

(ii). It is proved analogously to part (i).

(iii). Now suppose that  $T$  is epimorphism and take  $(\Phi, \Psi) \in \mathcal{Z}_t^{(\ell)}((\mathcal{A} \times_T \mathcal{B})'')$ . By part (i),  $\Psi \in \mathcal{Z}_t^{(\ell)}(\mathcal{B}'')$ . We show that  $\Phi \in \mathcal{Z}_t^{(\ell)}(\mathcal{A}'')$ . As we mentioned before,  $T'' : \mathcal{B}'' \rightarrow \mathcal{A}''$  is epimorphism, as well. Thus for each  $\Phi' \in \mathcal{A}''$ , there is  $\Psi_0 \in \mathcal{B}''$  such that  $T''(\Psi_0) = \Phi'$ . It follows that

$$T''(\Psi) \square \Phi' = T''(\Psi) \square T''(\Psi_0) = T''(\Psi \diamond \Psi_0) = T''(\Psi) \diamond \Phi'. \quad (3.4)$$

This equality together with (3.3) yield that

$$\Phi \square \Phi' = \Phi \diamond \Phi'.$$

It follows that  $\Phi \in \mathcal{Z}_t^{(\ell)}(\mathcal{A}'')$ . Therefore

$$\mathcal{Z}_t^{(\ell)}((\mathcal{A} \times_T \mathcal{B})'') \subseteq \mathcal{Z}_t^{(\ell)}(\mathcal{A}'') \times_{T''} \mathcal{Z}_t^{(\ell)}(\mathcal{B}'').$$

The reverse of the above inclusion is also established. Indeed, suppose that  $(\Phi, \Psi) \in \mathcal{Z}_t^{(\ell)}(\mathcal{A}'') \times_{T''} \mathcal{Z}_t^{(\ell)}(\mathcal{B}'')$ . By (3.4), for all  $(\Phi', \Psi') \in (\mathcal{A} \times_T \mathcal{B})''$  we easily obtain

$$(\Phi, \Psi) \square (\Phi', \Psi') = (\Phi, \Psi) \diamond (\Phi', \Psi'),$$

that is,  $(\Phi, \Psi) \in \mathcal{Z}_t^{(\ell)}((\mathcal{A} \times_T \mathcal{B})'')$ . Thus

$$\mathcal{Z}_t^{(\ell)}(\mathcal{A}'') \times_{T''} \mathcal{Z}_t^{(\ell)}(\mathcal{B}'') \subseteq \mathcal{Z}_t^{(\ell)}((\mathcal{A} \times_T \mathcal{B})'')$$

and therefore

$$\mathcal{Z}_t^{(\ell)}((\mathcal{A} \times_T \mathcal{B})'') = \mathcal{Z}_t^{(\ell)}(\mathcal{A}'') \times_{T''} \mathcal{Z}_t^{(\ell)}(\mathcal{B}''),$$

as claimed. It follows that  $\mathcal{A} \times_T \mathcal{B}$  is Arens regular if and only if both  $\mathcal{A}$  and  $\mathcal{B}$  are Arens regular.  $\square$

#### 4. Weak amenability and character amenability

Let  $\mathcal{A}$  be a Banach algebra, and let  $X$  be a Banach  $\mathcal{A}$ -bimodule. A linear map  $D : \mathcal{A} \rightarrow X$  is called a derivation if  $D(ab) = D(a) \cdot b + a \cdot D(b)$ , for all  $a, b \in \mathcal{A}$ . Given  $x \in X$ , let  $ad_x : \mathcal{A} \rightarrow X$  be given by  $ad_x(a) = a \cdot x - x \cdot a$  ( $a \in \mathcal{A}$ ). Then  $ad_x$  is a derivation which is called an inner derivation at  $x$ . The Banach algebra  $\mathcal{A}$  is called weakly amenable if and only if every continuous derivation  $D : \mathcal{A} \rightarrow \mathcal{A}'$  is inner. If  $I$  is a closed ideal of  $\mathcal{A}$ , then, by [3, Proposition 2.8.66],  $\mathcal{A}$  is weakly amenable if  $I$  and  $\mathcal{A}/I$  are weakly amenable. As the first result of this section, we prove [2, Theorem 4.1, part (2)] for the case where  $\mathcal{A}$  is not necessarily commutative.

**Theorem 4.1.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be Banach algebras and  $T \in \text{hom}(\mathcal{B}, \mathcal{A})$ . Then  $\mathcal{A} \times_T \mathcal{B}$  is weakly amenable if and only if  $\mathcal{A}$  and  $\mathcal{B}$  are weakly amenable.*

*Proof.* We only explain the proof of [2, Theorem 4.1, Part (2)], with respect to definition (1.1), and for the case where  $\mathcal{A}$  is not necessarily commutative. Assume that both  $\mathcal{A}$  and  $\mathcal{B}$  are weakly amenable. Since  $\mathcal{A}$  is a closed ideal in  $\mathcal{A} \times_T \mathcal{B}$  and  $(\mathcal{A} \times_T \mathcal{B})/\mathcal{A} \cong \mathcal{B}$ , thus  $\mathcal{A} \times_T \mathcal{B}$  is weakly amenable, by [3, Proposition 2.8.66]. Conversely, let  $\mathcal{A} \times_T \mathcal{B}$  be weakly amenable and  $d_1 : \mathcal{A} \rightarrow \mathcal{A}'$  and  $d_2 : \mathcal{B} \rightarrow \mathcal{B}'$  be continuous derivations. Moreover suppose that  $P_1 : \mathcal{A} \times_T \mathcal{B} \rightarrow \mathcal{A}$  and  $P_2 : \mathcal{A} \times_T \mathcal{B} \rightarrow \mathcal{B}$  are defined as  $P_1(a, b) = a + T(b)$  and  $P_2(a, b) = b$ . Also let  $D_1 = P_1' \circ d_1 \circ P_1$  and  $D_2 = P_2' \circ d_2 \circ P_2$ , be as in the proof of [2, Theorem 4.1, Part (2)]. Then  $D_1, D_2 : \mathcal{A} \times_T \mathcal{B} \rightarrow \mathcal{A}' \times \mathcal{B}'$  are continuous derivations and since  $\mathcal{A} \times_T \mathcal{B}$  is weakly amenable, there exist  $(\varphi_1, \psi_1)$  and  $(\varphi_2, \psi_2)$  in  $\mathcal{A}' \times_T \mathcal{B}'$  such that  $D_1 = ad_{(\varphi_1, \psi_1)}$  and  $D_2 = ad_{(\varphi_2, \psi_2)}$ . Similar arguments to the proof of [2, Theorem 4.1, Part (2)] imply that  $d_1 = ad_{\varphi_1}$  and  $d_2 = ad_{\psi_2}$ . It follows that  $\mathcal{A}$  and  $\mathcal{B}$  are weakly amenable.  $\square$

Let  $\sigma(\mathcal{A})$  be the character space of  $\mathcal{A}$ , the space consisting of all non-zero continuous multiplicative linear functionals on  $\mathcal{A}$ . In [2, Theorem 2.1],  $\sigma(\mathcal{A} \times_T \mathcal{B})$  has been characterized, for the case where  $\mathcal{A}$  is commutative. The arguments, used in the proof of [2, Theorem 2.1] will be worked for the case where we use the definition (1.1) for  $\times_T$ , and also  $\mathcal{A}$  is not commutative. In fact

$$\sigma(\mathcal{A} \times_T \mathcal{B}) = \{(\varphi, \varphi \circ T, ) : \varphi \in \sigma(\mathcal{A})\} \cup \{(0, \psi) : \psi \in \sigma(\mathcal{B})\},$$

as a disjoint union.

Following [7], a Banach algebra  $\mathcal{A}$  is called left character amenable if for all  $\psi \in \sigma(\mathcal{A}) \cup \{0\}$  and all Banach  $\mathcal{A}$ -bimodules  $E$  for which the right module action is given by  $x \cdot a = \psi(a)x$  ( $a \in \mathcal{A}, x \in E$ ), every continuous derivation  $d : \mathcal{A} \rightarrow E$  is inner. Right character amenability is defined similarly by considering Banach  $\mathcal{A}$ -bimodules  $E$  for which the left module action is given by  $a \cdot x = \psi(a)x$  ( $a \in \mathcal{A}, x \in E$ ). In this section we study left character amenability of  $\mathcal{A} \times_T \mathcal{B}$ . Before, we investigate [7, Proposition 2.8] for  $\mathcal{A} \times_T \mathcal{B}$ , which is useful for our purpose. Recall from [7] that, for  $\varphi \in \sigma(\mathcal{A}) \cup \{0\}$  and  $\Phi \in \mathcal{A}''$ ,  $\Phi$  is called  $\varphi$ -topologically left invariant ( $\varphi$ -TLI) if

$$\langle \Phi, a \cdot f \rangle = \varphi(a) \langle \Phi, f \rangle \quad (a \in \mathcal{A}, f \in \mathcal{A}'),$$

or equivalently  $\Phi \square a = \varphi(a)\Phi$ . Also  $\Phi$  is called  $\varphi$ -topologically right invariant ( $\varphi$ -TRI) if

$$\langle \Phi, f \cdot a \rangle = \varphi(a) \langle \Phi, f \rangle \quad (a \in \mathcal{A}, f \in \mathcal{A}'),$$

or equivalently  $a \square \Phi = \varphi(a)\Phi$ .

**Theorem 4.2.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be Banach algebras,  $T \in \text{hom}(\mathcal{B}, \mathcal{A})$  and  $(\Phi, \Psi) \in (\mathcal{A} \times_T \mathcal{B})'' = \mathcal{A}'' \times_{T''} \mathcal{B}''$ .*

- (i) *For  $\varphi \in \sigma(\mathcal{A})$ ,  $(\Phi, \Psi)$  is  $(\varphi, \varphi \circ T)$ -TLI with  $\langle (\Phi, \Psi), (\varphi, \varphi \circ T) \rangle \neq 0$  if and only if  $\Psi = 0$  and  $\Phi$  is  $\varphi$ -TLI with  $\Phi(\varphi) \neq 0$ .*
- (ii) *For  $\psi \in \sigma(\mathcal{B})$ ,  $(\Phi, \Psi)$  is  $(0, \psi)$ -TLI with  $\langle (\Phi, \Psi), (0, \psi) \rangle \neq 0$  if and only if  $\Psi$  is  $\psi$ -TLI with  $\Psi(\psi) \neq 0$  and  $\Phi = -T''(\Psi)$ .*

*Similar results hold for topologically right invariant elements.*

*Proof.* (i) Let  $(\Phi, \Psi)$  be  $(\varphi, \varphi \circ T)$ -TLI with  $\langle (\Phi, \Psi), (\varphi, \varphi \circ T) \rangle \neq 0$ . Thus for all  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$  we have

$$(\Phi, \Psi) \square(a, b) = (\varphi(a) + (\varphi \circ T)(b))(\Phi, \Psi)$$

and

$$\langle (\Phi, \Psi), (\varphi, \varphi \circ T) \rangle = \Phi(\varphi) + \Psi(\varphi \circ T) \neq 0. \quad (4.1)$$

Consequently

$$(\Phi, \Psi) \square(a, b) = (\Phi \square a + \Phi \square T(b) + T''(\Psi) \square a, \Psi \square b) = (\varphi(a) + (\varphi \circ T)(b))(\Phi, \Psi).$$

It follows that

$$\Phi \square a + \Phi \square T(b) + T''(\Psi) \square a = \varphi(a)\Phi + (\varphi \circ T)(b)\Phi \quad (4.2)$$

and

$$\Psi \square b = \varphi(a)\Psi + (\varphi \circ T)(b)\Psi. \quad (4.3)$$

Choosing  $b = 0$  and  $a \in \mathcal{A}$  with  $\varphi(a) \neq 0$ , we conclude from (4.3) that  $\varphi(a)\Psi = 0$ , which implies  $\Psi = 0$ . Also by (4.2) we obtain  $\Phi \square a = \varphi(a)\Phi$ . On the other hand since  $\Psi = 0$ , by (4.1) we get  $\Phi(\varphi) \neq 0$ . Consequently  $\Phi$  is  $\varphi$ -TLI with  $\Phi(\varphi) \neq 0$ .

For the converse, suppose that  $\Phi$  is  $\varphi$ -TLI with  $\Phi(\varphi) \neq 0$ . Thus

$$\langle (\Phi, 0), (\varphi, \varphi \circ T) \rangle = \Phi(\varphi) \neq 0.$$

Also for all  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$ ,

$$(\Phi, 0) \square(a, b) = (\Phi \square a + \Phi \square T(b), 0).$$

On the other hand

$$(\varphi(a) + (\varphi \circ T)(b))(\Phi, 0) = (\varphi(a)\Phi + (\varphi \circ T)(b)\Phi, 0).$$

By the hypothesis,  $\Phi \square a = \varphi(a)\Phi$  and  $\Phi \square T(b) = (\varphi \circ T)(b)\Phi$ . Thus

$$(\Phi, 0) \square(a, b) = (\varphi(a) + (\varphi \circ T)(b))(\Phi, 0).$$

Consequently  $(\Phi, 0)$  is  $(\varphi, \varphi \circ T)$ -TLI with  $\langle (\Phi, 0), (\varphi, \varphi \circ T) \rangle = \Phi(\varphi) \neq 0$ .

(ii) Let  $(\Phi, \Psi)$  be  $(0, \psi)$ -TLI with  $\langle (\Phi, \Psi), (0, \psi) \rangle \neq 0$ . It follows that  $\Psi(\psi) \neq 0$ . Moreover, for all  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$  we have

$$\begin{aligned} (\Phi \square a + \Phi \square T(b) + T''(\Psi) \square a, \Psi \square b) &= (\Phi, \Psi) \square(a, b) \\ &= (\psi(b))(\Phi, \Psi) \\ &= (\psi(b)\Phi, \psi(b)\Psi). \end{aligned}$$

So  $\Psi \square b = \psi(b)\Psi$  and consequently  $\Psi$  is  $\psi$ -TLI with  $\Psi(\psi) \neq 0$ . Furthermore

$$\Phi \square a + \Phi \square T(b) + T''(\Psi) \square a = \psi(b)\Phi. \quad (4.4)$$

Choosing  $a = 0$  in (4.4), we obtain

$$\Phi \square T(b) = \psi(b)\Phi \quad (b \in \mathcal{B}) \quad (4.5)$$

and by choosing  $b = 0$  in (4.4) we get

$$\Phi \square a + T''(\Psi) \square a = 0 \quad (a \in \mathcal{A}). \quad (4.6)$$

On the other hand for all  $b \in \mathcal{B}$

$$T''(\Psi) \square T(b) = T''(\Psi \square b) = T''(\psi(b)\Psi) = \psi(b)T''(\Psi). \quad (4.7)$$

Suppose that  $b \in \mathcal{B}$  with  $\psi(b) \neq 0$ . Using (4.6) for  $a = T(b)$ , and also (4.5) and (4.7) we obtain

$$0 = \Phi \square T(b) + T''(\Psi) \square T(b) = \psi(b)\Phi + \psi(b)T''(\Psi),$$

which implies  $\Phi + T''(\Psi) = 0$  and so  $\Phi = -T''(\Psi)$ .

For the converse, suppose that  $\Psi$  is  $\psi$ -TLI with  $\Psi(\psi) \neq 0$ . It follows that

$$\langle (-T''(\Psi), \Psi), (0, \psi) \rangle = \Psi(\psi) \neq 0.$$

We show that  $(-T''(\Psi), \Psi)$  is  $(0, \psi)$ -TLI. For all  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$  we have

$$(-T''(\Psi), \Psi) \square (a, b) = (-T''(\Psi) \square T(b), \Psi \square b).$$

On the other hand

$$\psi(b)(-T''(\Psi), \Psi) = (-\psi(b)T''(\Psi), \psi(b)\Psi).$$

Since  $\Psi$  is  $\psi$ -TLI, thus  $\Psi \square b = \psi(b)\Psi$  and consequently  $T''(\Psi) \square T(b) = \psi(b)T''(\Psi)$ . It follows that

$$(-T''(\Psi), \Psi) \square (a, b) = \psi(b)(-T''(\Psi), \Psi),$$

which implies that  $(-T''(\Psi), \Psi)$  is  $(0, \psi)$ -TLI.  $\square$

In the sequel, we prove that  $\mathcal{A} \times_T \mathcal{B}$  is left (right) character amenable if and only if both  $\mathcal{A}$  and  $\mathcal{B}$  so are. In fact we prove the result provided for the  $\theta$ -Lau product given in [7, Corollary 2.9], for the morphism product  $\mathcal{A} \times_T \mathcal{B}$ . Before, we recall some earlier results related to character amenability, which are useful for our purpose.

Let  $I$  be a closed two-sided ideal in the Banach algebra  $\mathcal{A}$ . By [7, Theorem 2.6], if both  $I$  and  $\mathcal{A}/I$  are left character amenable, then  $\mathcal{A}$  is also left character amenable. Also by [7, Theorem 2.3],  $\mathcal{A}$  is left character amenable if and only if the following two conditions hold:

- (i)  $\mathcal{A}$  has a bounded left approximate identity,
- (ii) for every  $\psi \in \sigma(\mathcal{A})$ , there exists a  $\psi$ -TLI element  $\Psi \in \mathcal{A}''$  such that  $\Psi(\psi) \neq 0$ .

Similar statements hold for right character amenability.

**Theorem 4.3.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be Banach algebras and  $T \in \text{hom}(\mathcal{B}, \mathcal{A})$ . Then  $\mathcal{A} \times_T \mathcal{B}$  is left (right) character amenable if and only if both  $\mathcal{A}$  and  $\mathcal{B}$  are left (right) character amenable.*

*Proof.* We only prove the left version. Suppose that  $\mathcal{A}$  and  $\mathcal{B}$  are left character amenable. Since  $\mathcal{A}$  is an ideal in  $\mathcal{A} \times_T \mathcal{B}$  and  $(\mathcal{A} \times_T \mathcal{B})/\mathcal{A} \cong \mathcal{B}$ , thus  $\mathcal{A} \times_T \mathcal{B}$  is also left character amenable, by [7, Theorem 2.6]. Conversely, suppose that  $\mathcal{A} \times_T \mathcal{B}$  is left character amenable. Then the map

$$\mu : \mathcal{A} \times_T \mathcal{B} \rightarrow \mathcal{B}; \quad (a, b) \mapsto b,$$



is clearly a continuous epimorphism. Now [7, Theorem 2.6] implies that  $\mathcal{B}$  is also left character amenable. Now we show that  $\mathcal{A}$  is left character amenable. Since  $\mathcal{A} \times_T \mathcal{B}$  is left character amenable, then by [7, Theorem 2.3],  $\mathcal{A} \times_T \mathcal{B}$  has a bounded left approximate identity. So by [1, Proposition 3.2],  $\mathcal{A}$  has a bounded left approximate identity. Moreover, by the hypothesis and also [7, Theorem 2.3], for each  $\varphi \in \sigma(\mathcal{A})$ , there exists a  $(\varphi, \varphi \circ T)$ -TLI element  $(\Phi, \Psi) \in \mathcal{A}'' \times_{T''} \mathcal{B}''$  such that

$$\Phi(\varphi) + \Psi(\varphi \circ T) \neq 0.$$

By part (i) of Theorem 4.2, we have  $\Psi = 0$  and  $\Phi$  is a  $\varphi$ -TLI element with  $\Phi(\varphi) \neq 0$ . Therefore  $\mathcal{A}$  is left character amenable again by [7, Theorem 2.3].  $\square$

## 5. Character inner amenability

Let  $\mathcal{A}$  be a Banach algebra and  $\varphi \in \sigma(\mathcal{A})$ . Following [4],  $\mathcal{A}$  is  $\varphi$ -inner amenable if there exists  $m \in \mathcal{A}''$  such that  $m(\varphi) = 1$  and  $m \square a = a \square m$ , for all  $a \in \mathcal{A}$ . Such an  $m$  is called a  $\varphi$ -inner mean for  $\mathcal{A}''$ . A Banach algebra  $\mathcal{A}$  is called character inner amenable if  $\mathcal{A}$  is  $\varphi$ -inner amenable, for all  $\varphi \in \sigma(\mathcal{A})$ . The aim of the present section is to prove that character inner amenability of  $\mathcal{A} \times_T \mathcal{B}$  is stable with respect to  $T$ . In fact we generalize [2, Theorem 4.2, part (3)] with respect to the definition (1.1). We commence with following result.

**Theorem 5.1.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be Banach algebras,  $T \in \text{hom}(\mathcal{B}, \mathcal{A})$  and  $\varphi \in \sigma(\mathcal{A})$ . Then  $\mathcal{A}$  is  $\varphi$ -inner amenable if and only if  $\mathcal{A} \times_T \mathcal{B}$  is  $(\varphi, \varphi \circ T)$ -inner amenable.*

*Proof.* First let  $\mathcal{A}$  be  $\varphi$ -inner amenable. So there exists  $m \in \mathcal{A}''$  such that  $m(\varphi) = 1$  and

$$a \square m = m \square a,$$

for each  $a \in \mathcal{A}$ . It follows that

$$\langle (m, 0), (\varphi, \varphi \circ T) \rangle = m(\varphi) = 1.$$

Moreover for each  $(a, b) \in \mathcal{A} \times_T \mathcal{B}$ ,

$$(a, b) \square (m, 0) = (a \square m + T(b) \square m, 0)$$

and

$$(m, 0) \square (a, b) = (m \square a + m \square T(b), 0).$$

Since  $a \square m = m \square a$  and  $T(b) \square m = m \square T(b)$ , it follows that

$$(a, b) \square (m, 0) = (m, 0) \square (a, b).$$

Thus  $(m, 0)$  is a  $(\varphi, \varphi \circ T)$ -inner mean for  $\mathcal{A} \times_T \mathcal{B}$  and so  $(\varphi, \varphi \circ T)$ -inner amenability of  $\mathcal{A} \times_T \mathcal{B}$  is obtained.

Conversely suppose that  $\mathcal{A} \times_T \mathcal{B}$  is  $(\varphi, \varphi \circ T)$ -inner amenable. Thus there exists  $(m, n) \in \mathcal{A}'' \times_{T''} \mathcal{B}''$  such that

$$\langle (m, n), (\varphi, \varphi \circ T) \rangle = 1 \tag{5.1}$$

and for all  $(a, b) \in \mathcal{A} \times_T \mathcal{B}$

$$(m, n) \square (a, b) = (a, b) \square (m, n).$$

So we obtain

$$(m \square a + m \square T(b) + T''(n) \square a, n \square b) = (a \square m + a \square T''(n) + T(b) \square m, b \square n).$$

It follows that for all  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$ ,

$$m \square a + m \square T(b) + T''(n) \square a = a \square m + a \square T''(n) + T(b) \square m.$$

Choosing  $b = 0$ , we obtain

$$m \square a + T''(n) \square a = a \square m + a \square T''(n),$$

or equivalently

$$(m + T''(n)) \square a = a \square (m + T''(n)),$$

for each  $a \in \mathcal{A}$ . On the other hand by (5.1) we have

$$\begin{aligned} \langle m + T''(n), \varphi \rangle &= \langle m, \varphi \rangle + \langle T''(n), \varphi \rangle \\ &= \langle m, \varphi \rangle + \langle n, T'(\varphi) \rangle \\ &= \langle m, \varphi \rangle + \langle n, \varphi \circ T \rangle \\ &= 1. \end{aligned}$$

Consequently  $m + T''(n)$  is a  $\varphi$ -inner mean for  $\mathcal{A}$  and therefore  $\mathcal{A}$  is  $\varphi$ -inner amenable.  $\square$

**Theorem 5.2.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be Banach algebras,  $T \in \text{hom}(\mathcal{B}, \mathcal{A})$  and  $\varphi \in \sigma(\mathcal{A})$ . If  $(m, n)$  is a  $(\varphi, \varphi \circ T)$ -inner mean for  $\mathcal{A} \times_T \mathcal{B}$  and  $n(\varphi \circ T) \neq 0$ , then  $\mathcal{B}$  is  $\varphi \circ T$ -inner amenable. Moreover if  $T$  is epimorphism and  $\mathcal{B}$  is  $\varphi \circ T$ -inner amenable, then  $\mathcal{A} \times_T \mathcal{B}$  is  $(\varphi, \varphi \circ T)$ -inner amenable.*

*Proof.* Suppose that  $(m, n)$  is a  $(\varphi, \varphi \circ T)$ -inner mean for  $\mathcal{A} \times_T \mathcal{B}$  such that  $n(\varphi \circ T) \neq 0$ . Thus  $\langle (m, n), (\varphi, \varphi \circ T) \rangle = 1$  and

$$(m, n) \square (a, b) = (a, b) \square (m, n) \quad ((a, b) \in \mathcal{A} \times_T \mathcal{B}).$$

Consequently we obtain

$$(m \square a + m \square T(b) + T''(n) \square a, n \square b) = (a \square m + a \square T''(n) + T(b) \square m, b \square n).$$

It follows that  $n \square b = b \square n$ , for all  $b \in \mathcal{B}$ . Since  $n(\varphi \circ T) \neq 0$ , it follows that

$$\left( \frac{n}{n(\varphi \circ T)} \right) (\varphi \circ T) = 1.$$

Moreover for each  $b \in \mathcal{B}$ , we have

$$b \square \left( \frac{n}{n(\varphi \circ T)} \right) = \left( \frac{n}{n(\varphi \circ T)} \right) \square b.$$

It follows that  $\frac{n}{n(\varphi \circ T)}$  is a  $(\varphi \circ T)$ -inner mean for  $\mathcal{B}$ , which implies that  $\mathcal{B}$  is  $(\varphi \circ T)$ -inner amenable.

Now suppose that  $T$  is epimorphism and  $\mathcal{B}$  is  $\varphi \circ T$ -inner amenable. Thus there exists  $n \in \mathcal{B}''$  such that  $n(\varphi \circ T) = 1$  and  $n \square b = b \square n$ , for each  $b \in \mathcal{B}$ . Take  $(0, n) \in \mathcal{A}'' \times_{T''} \mathcal{B}''$ . Thus

$$\langle (0, n), (\varphi, \varphi \circ T) \rangle = n(\varphi \circ T) = 1.$$

On the other hand, for each  $(a, b) \in \mathcal{A} \times_T \mathcal{B}$ , we have

$$\begin{cases} (0, n) \square (a, b) = (T''(n) \square a, n \square b), \\ (a, b) \square (0, n) = (a \square T''(n), b \square n). \end{cases}$$

Since  $T$  is onto, for each  $a \in \mathcal{A}$  there exists  $b' \in \mathcal{B}$  such that  $T(b') = a$ . Consequently

$$T''(n) \square a = T''(n) \square T(b') = T''(n \square b') = T''(b' \square n) = T(b') \square T''(n) = a \square T''(n).$$

These observations show that

$$(0, n) \square (a, b) = (a, b) \square (0, n).$$

Thus  $(0, n)$  is a  $(\varphi, \varphi \circ T)$ -inner mean for  $\mathcal{A} \times_T \mathcal{B}$  and therefore  $\mathcal{A} \times_T \mathcal{B}$  is  $(\varphi, \varphi \circ T)$ -inner amenable.  $\square$

**Theorem 5.3.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be Banach algebras,  $T \in \text{hom}(\mathcal{B}, \mathcal{A})$  and  $\psi \in \sigma(\mathcal{B})$ . Then  $\mathcal{A} \times_T \mathcal{B}$  is  $(0, \psi)$ -inner amenable if and only if  $\mathcal{B}$  is  $\psi$ -inner amenable.*

*Proof.* Assume that  $\mathcal{A} \times_T \mathcal{B}$  is  $(0, \psi)$ -inner amenable. Thus there exists  $(m, n) \in \mathcal{A}'' \times_{T''} \mathcal{B}''$  such that

$$\langle (m, n), (0, \psi) \rangle = n(\psi) = 1$$

and for each  $(a, b) \in \mathcal{A} \times_T \mathcal{B}$ ,

$$(a, b) \square (m, n) = (m, n) \square (a, b).$$

Consequently  $b \square n = n \square b$ , for each  $b \in \mathcal{B}$ . It follows that  $n$  is a  $\psi$ -inner mean for  $\mathcal{B}$  and so  $\mathcal{B}$  is  $\psi$ -inner amenable. Conversely, suppose that  $\mathcal{B}$  is  $\psi$ -inner amenable. Thus there exists  $n \in \mathcal{B}''$  such that  $n(\psi) = 1$  and  $b \square n = n \square b$ , for all  $b \in \mathcal{B}$ . Take  $(-T''(n), n) \in \mathcal{A}'' \times_{T''} \mathcal{B}''$ . Thus

$$\langle (-T''(n), n), (0, \psi) \rangle = n(\psi) = 1.$$

Moreover for all  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$

$$\begin{aligned} (a, b) \square (-T''(n), n) &= (-a \square T''(n) + a \square T''(n) - T(b) \square T''(n), b \square n) \\ &= (-T(b) \square T''(n), b \square n), \end{aligned}$$

and

$$\begin{aligned} (-T''(n), n) \square (a, b) &= (-T''(n) \square a - T''(n) \square T(b) + T''(n) \square a, n \square b) \\ &= (-T''(n) \square T(b), n \square b). \end{aligned}$$

Note that

$$T(b) \square T''(n) = T''(b \square n) = T''(n \square b) = T''(n) \square T(b).$$

Consequently

$$(a, b) \square (-T''(n), n) = (-T''(n), n) \square (a, b).$$

Thus  $(-T''(n), n)$  is a  $(0, \psi)$ -inner mean for  $\mathcal{A} \times_T \mathcal{B}$ , which implies that  $\mathcal{A} \times_T \mathcal{B}$  is  $(0, \psi)$ -inner amenable, as claimed.  $\square$

In [2, Theorem 4.2, Part (3)], it has been proved that  $\mathcal{A} \times_T \mathcal{B}$  is character inner amenable if and only if  $\mathcal{B}$  is character inner amenable. In fact since  $\mathcal{A}$  is assumed to be commutative, thus  $\mathcal{A}$  is spontaneously character inner amenable. Note that in the proof of this result, the identification  $(\mathcal{A} \times_T \mathcal{B}) \cong \mathcal{A}'' \times_{T''} \mathcal{B}''$  is used. Thus by [2, Theorem 3.1], in the assumption of [2, Theorem 4.2, Part (3)] in fact  $\mathcal{A}$  should be commutative and Arens regular. We state here the main result of the present section, which is a generalization of [2, Theorem 4.2, Part (3)], with respect to the definition (1.1) and for an arbitrary Banach algebra  $\mathcal{A}$ .

**Theorem 5.4.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be Banach algebras and  $T \in \text{hom}(\mathcal{B}, \mathcal{A})$ . Then  $\mathcal{A} \times_T \mathcal{B}$  is character inner amenable if and only if both  $\mathcal{A}$  and  $\mathcal{B}$  are character inner amenable.*

*Proof.* Assume that  $\mathcal{A} \times_T \mathcal{B}$  is character inner amenable. Thus  $\mathcal{A} \times_T \mathcal{B}$  is  $(\varphi, \varphi \circ T)$ -inner amenable, for all  $\varphi \in \sigma(\mathcal{A})$ . By Theorem 5.1,  $\mathcal{A}$  is  $\varphi$ -inner amenable, for all  $\varphi \in \sigma(\mathcal{A})$  and consequently  $\mathcal{A}$  is character inner amenable. Now suppose that  $\psi \in \sigma(\mathcal{B})$ . By the hypothesis,  $\mathcal{A} \times_T \mathcal{B}$  is  $(0, \psi)$ -inner amenable and so  $\mathcal{B}$  is  $\psi$ -inner amenable, by Theorem 5.3. It follows that  $\mathcal{B}$  is character inner amenable.

Conversely, suppose that  $\mathcal{A}$  and  $\mathcal{B}$  are character inner amenable. Thus  $\mathcal{A}$  is  $\varphi$ -inner amenable, for all  $\varphi \in \sigma(\mathcal{A})$  and by Theorem 5.1,  $\mathcal{A} \times_T \mathcal{B}$  is  $(\varphi, \varphi \circ T)$ -inner amenable. Moreover  $\mathcal{B}$  is  $\psi$ -inner amenable, for all  $\psi \in \sigma(\mathcal{B})$ . Thus by Theorem 5.3,  $\mathcal{A} \times_T \mathcal{B}$  is  $(0, \psi)$ -inner amenable. Therefore  $\mathcal{A} \times_T \mathcal{B}$  is character inner amenable.  $\square$

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